Alicia Mae Lamarche *Research Statement*

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I am an algebraic geometer who is interested in using the machinery of derived categories to answer questions of number theoretic origin. The main focus of my dissertation investigates how the rationality of a smooth projective variety X manifests itself inside of its associated derived category of coherent sheaves, henceforth denoted $\mathrm{D^b}(X)$. I am excited to continue research in this direction, and I am broadly interested in the intersection of birational geometry and arithmetic geometry as well as combinatorial/computational methods to approaching these subjects.

1 Introduction: Rationality Questions & Motivation

Determining the existence of solutions to a system of equations is a fundamental problem in mathematics that has historically provided mathematicians with a wealth of surprisingly difficult phenomena to ponder. At the heart of this is the following question: when does a system of polynomials with integer coefficients have integer solutions?

A budding mathematician may learn in an elementary number theory class that the linear equation $ax + by = c$ for $a, b, c \in \mathbb{Z}$ has a solution if and only if $\gcd(a, b)$ divides c (given that $a, b \neq 0$). For quadratics, the *Hasse-Minkowski Theorem* tells us that integral solutions exist if and only if there exist solutions over every completion of $\mathbb{Q}$; that is, there exist real solutions and solutions in $\mathbb{Q}_p$ for every prime p. This is an example of what is now known as the *Hasse principle*, the ability to stitch together local solutions, e.g. solutions over $\mathbb{Q}_p$, to get a global solution, e.g. a solution over $\mathbb{Q}$. Although this is a desirable property to have, it does not always hold. For example, consider $x^4 - 17 = 2y^2$, which has solutions in $\mathbb{R}$ and over $\mathbb{Q}_p$ for all primes p, but no solutions over $\mathbb{Q}$.

Algebraic geometers are also interested in solutions sets to systems of polynomials, although they are frequently studied under the guise of *algebraic varieties*. The *sheaf cohomology* of a variety is in many ways a geometric analogue of the Hasse principal: it measures the failure of the ability to glue together local solutions to a geometric problem to obtain a global solution over the entire variety. If we are looking for obstructions to rational points, it is therefore natural to turn to a well-studied repository of the cohomological information attached to a variety X, $\mathrm{D^b}(X)$- its derived category of coherent sheaves. This leads one to wonder the following: over a non-algebraically closed field k, what sort of arithmetic properties does the variety X enjoy, and can these be detected by the derived category of coherent sheaves on X? In broad strokes, my interests lie in investigating arithmetic properties revealed via the derived category of a variety. One property that can be studied is varying strengths of *rationality*. We say a variety X is *rational* if it is birational to projective space over some field.

Example 1. Consider first the rational variety $\mathbb{P}^1_{\mathbb{C}}$. Let $X = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{P}^2_{\mathbb{R}}$, and notice that $X \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{P}^1_{\mathbb{C}}$. We say that X is a *form* of $\mathbb{P}^1_{\mathbb{C}}$, that is, it becomes isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ when base-changed to $\mathbb{C}$, in fact: X is a *Severi-Brauer variety*- a "twisted form" of projective space. Over $\mathbb{R}$, X is not rational, as it is not birational to $\mathbb{P}^1_{\mathbb{R}}$. A quick way to convince yourself of this is to note that $\mathbb{P}^1_{\mathbb{R}}$ admits many $\mathbb{R}$-points, but X does not.

2 Arithmetic & the derived category

During my time as a graduate student, my main focus was on the study of decompositions of the derived category of coherent sheaves of a variety. For a smooth projective variety X defined over a field k, we call this a *semi-orthogonal decomposition.* Originally described in [14], the existence of a *full exceptional collection* of $\mathrm{D^b}(\mathsf{Coh}(X))$, a particularly well-behaved example of a semi-orthogonal decomposition, provides us with an atomization that, in many ways, mimics an orthogonal decomposition of a vector space. Exceptional collections are built of *exceptional objects*, which are objects of a k-linear category whose endomorphism algebras are isomorphic to division k-algebras. Due to recent work of Ballard, Duncan, and McFaddin [6] on derived categories of forms toric varieties over an arbitrary field, a large portion of my research program consists of utilizing these decompositions to answer questions with number-theoretic origins.

2.1 Rationality and the derived category

My recent work is motivated by the following question (due to Esnault, 2009 [4]):

Question 1. *"Let S be a smooth projective variety over a field k. Can the bounded derived category $\mathrm{D^b}(S)$ of coherent sheaves detect the existence of a k-rational point on S?"*

I have produced both positive and negative evidence of the ability of $\mathrm{D^b}(S)$ to detect rational points; the answer to this question is an emphatic *sometimes.* Recently, Ballard, Duncan, and McFaddin [7] gave a Galois-stable decomposition for the derived category of any centrally symmetric toric Fano variety. The '*Galois-stable*' stipulation requires that the objects of the decomposition are permuted by the absolute Galois group, which has the advantage of allowing such a decomposition to work regardless of the field over which the variety is defined. In [8], Matthew Ballard and I use the decomposition from [7] along with the power of noncommutative motives [43, 45] as a universal additive invariant to generalize a result of Blunk [17]:

Theorem 1. *(Ballard, L-) If X is a generalized del Pezzo variety, (in the sense of Voskresenskii & Klyachko [47]) then X has a rational point if and only if $\mathrm{D^b}(X)$ admits a full étale exceptional collection.*

A *full étale exceptional collection* is an exceptional collection in which the endomorphism algebra of each exceptional object is a finite separable extension of k. This is particularly nice property to have, as it essentially says that the derived category is built from derived categories of smooth points. The family of *generalized del Pezzo varieties* consists of arithmetic toric varieties whose rationality is determined by the dense open torus T contained inside of it. In this particular case, two algebras govern the triviality of T-torsors, and also happen to show up as endomorphism algebras of exceptional objects.

Example 2. Recall from *Example 1* that $\mathbb{P}^1_{\mathbb{C}}$ is rational, while a *"twisted-form"* of $\mathbb{P}^1_{\mathbb{C}}$, $X = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{P}^2_{\mathbb{R}}$ is not. To illustrate how rationality is reflected in $\mathrm{D^b}(\mathbb{P}^1_{\mathbb{C}})$ and $\mathrm{D^b}(X)$, let us compare decompositions of these two categories. We know from [14] that $\mathrm{D^b}(\mathbb{P}^1_{\mathbb{C}})$ can be decomposed as $\langle \mathcal{O}, \mathcal{O}(1)\rangle$, and one can check that $\mathrm{D^b}(X) = \langle \mathcal{O}, \mathcal{F}\rangle$, where $\mathrm{End}(\mathcal{F}) \cong \mathbb{H}$, Hamilton's quaternion algebra.

$$\mathrm{D^b}(\mathbb{P}^1_{\mathbb{C}}) = \langle \mathcal{O}, \mathcal{O}(1)\rangle \qquad \textit{versus} \qquad \mathrm{D^b}(X) = \langle \mathcal{O}, \mathcal{F}\rangle$$

Notice that the endomorphism algebras of the exceptional objects forming the decomposition of $\mathrm{D^b}(\mathbb{P}^1_{\mathbb{C}})$ are copies of $\mathbb{C}$, but a similar statement cannot be made for the decomposition of $\mathrm{D^b}(X)$:

$\text{End}(\mathcal{F}) \cong \mathbb{H}$ is not a copy of $\mathbb{R}$, nor is $\mathbb{H}$ a separable extension of $\mathbb{R}$. Surprisingly, this reveals a significant amount of information about the rationality of the associated varieties. It is known [43] that non-split Severi-Brauer varieties cannot admit a full étale exceptional collection, from this we can determine from the decomposition of $\mathrm{D^b}(X)$ that X is not rational.

On the other hand, in a forthcoming collaboration with Matthew Ballard, Alex Duncan, Patrick McFaddin, and I [10] show the following:

Theorem 2. *(Ballard, Duncan, L-, McFaddin) For a fixed global field k, there exist smooth projective geometrically-irreducible varieties in any dimension ≥ 3 which have no rational points but whose derived category possesses a full étale exceptional collection.*

From *Theorem 1* and *Theorem 2* we know that even when a full étale exceptional collection exists for $\mathrm{D^b}(X)$, this does not imply that X is rational. In [10], we show that for toric varieties, the existence of a full étale exceptional collection is enough to guarantee retract rationality. One can ask what might happen if we impose more strict conditions on the decomposition of $\mathrm{D^b}(X)$, which gives a question commonly attributed to Orlov:

Question 2. *"If X is a smooth variety defined over a field k, and $\mathrm{D^b}(X)$ admits a full k-exceptional collection, must X be rational?"*

In [10], we give the following positive answer to Orlov's question for toric varieties.

Theorem 3. *(Ballard, Duncan, L-, McFaddin) For a toric variety X with a k-point, the existence of a full k-exceptional collection of $\mathrm{D^b}(X)$ guarantees that X is k-rational.*

2.2 Cohomological invariants

Theorem 2 motivated an investigation of what is and isn't able to be detected by cohomological invariants. Let X a smooth projective compactification of a linear algebraic group G that admits a full étale exceptional collection. This gives us the following:

$$H^1(k, G) \to H^2(k, D(K)), \tag{1}$$

with $D(K)$ the Cartier dual of $K_0(X_k)$, this is a *cohomological invariant.* We show that when X_k is twisted by a torsor, the exceptional collection of the twisted variety is étale precisely when the torsor is in the kernel of (1). We use this to construct toric varieties in [10] which can be twisted by a nontrivial torsor which forces there to be no k-rational points, but the associated derived category still admits a full étale exceptional collection.

The result from *Theorem 2* indicates that there is information that cohomological invariants such as (1) are unable to detect. In [9] we give an explicit characterization of torsors that are not 'seen' by any Brauer groups. This motivates the definition of the following:

$$Ж(k, G) := \mathrm{im}\left(H^1(k, C) \to H^1(k, G)\right) \tag{2}$$

where $C \to G$ is a coflasque resolution of G as in [23]. In [9] we show that $Ж(k, G)$ is trivial when G is retract rational, and is a stable-birational invariant of G. Over a global field, this is the Tate-Shafarevich group of G, which is classically used to measure the extent to which the Hasse principal fails for the algebraic group G.

2.3 Weak approximation and other arithmetic considerations

In the introduction, a vague description of the *Hasse principle* was given, and we define it more precisely here. For a smooth projective variety X over a number field k, we say that X *satisfies the Hasse principle* whenever the following implication holds:

$$X(k_\nu) \neq \emptyset \text{ for all places } \nu \text{ of } k \implies X(k) \neq \emptyset,$$

where $X(k_\nu)$ denotes the set of k_ν-points of X, and $X(k)$ denotes the set of k-points. Given that a variety X admits k-points, one can ask (by defining the appropriate notion of a metric/distance) how 'close' rational points can be to one another, and the quantity of such points. The notion of *weak-approximation* formalizes this: when $X(\mathbb{A}_k) \neq \emptyset$, we say that X satisfies *weak-approximation* if $X(k)$ is dense in $X(\mathbb{A}_k)$ [1] under the product topology, where $X(\mathbb{A}_k)$ denotes the set of adelic points of X. A natural question given the theme of preceding sections is whether or not these arithmetic properties can be seen by the derived category.

Recently, Addington, Antieau, Frei, and Honigs [2] constructed the first examples of derived equivalent abelian varieties over $\mathbb{Q}$ where one has a rational point and the other does not. They also discuss how a pair of hyperkahler 4-folds defined over $\mathbb{Q}$ gives a transcendental Brauer-Manin obstruction to the Hasse principal. In an ongoing project with Sachi Hashimoto, Katrina Honigs, and Isabel Vogt, we study two Calabi-Yau three-folds that are known to be derived equivalent over $\mathbb{C}$ but *not* birational. [29, 30] (this is important, as the existence of rational points is a birational invariant) Although it seems to be the case that both of these Calabi-Yau three-folds will have rational points over $\mathbb{Q}$, we believe that one has a transcendental obstruction to weak-approximation.

3 Decomposing the derived category over an arbitrary field

In general, constructing decompositions of the derived category of an arbitrary variety is difficult; work of Kawamata [32, 33] shows that over an algebraically closed field of characteristic zero, such a decomposition exists for the derived category of any toric variety. By leveraging recent results in the field (see for example [6, 7, 26]) with special properties coming from their construction, I am interested in constructing full exceptional collections for certain families of varieties defined over an arbitrary field.

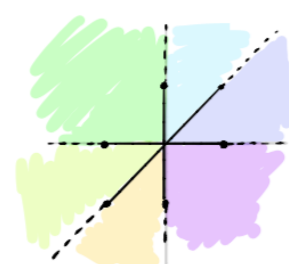

Figure 1: Fan of $X(\mathsf{A}_2)$

For any root system R, one can construct a toric variety $X(\mathsf{R})$ by defining the maximal cones of $\Sigma(\mathsf{R})$ to be generated by choices of simple roots of R, an example of such a fan $\Sigma(\mathsf{A}_2)$ corresponding to $X(\mathsf{A}_2)$ is shown to the right in *Figure 1.* (This is also the fan associated to the del Pezzo surface of degree six.) Recent work [6, 21] gives explicit Galois-stable semi-orthogonal decompositions for forms of $X(\mathsf{A}_\mathsf{n})$, albeit [21] approaches the problem from a moduli-theoretic perspective.

Example. There are many classical results coming from representation theory that can aid in the construction of Galois-stable exceptional collections. As an example, for $\mathrm{D^b}(X(\mathsf{C}_\mathsf{n}))$, (where $X(\mathsf{C}_\mathsf{n})$ is the toric variety associated to the root system of type C_n) it ought to be sufficient to collect objects in the decomposition of $\mathrm{D^b}(X(\mathsf{A}_{\mathsf{2n+1}}))$ that are invariant under involution. This is due to *folding*, which, in terms of root systems, is the process of quotienting out by a symmetry.

Toric varieties associated to root systems provide an important tool when studying generalizations of Losev-Manin moduli spaces [3, 12, 13, 41] and spherical varieties [18, 26]. These are all

[1] Note: because X is projective, we have that $X(\mathbb{A}_k) = \prod_\nu X(k_\nu)$.

incredibly rich theoretical objects that have found use within other disciplines such as mathematical physics [15].

4 *Future Directions:* Rationality & Arithmetic

The mechanism for using the derived category to determine rationality/arithmetic properties enjoyed by a variety is still not well-understood (for example, see [4,5,36]). From my perspective, the many recent advances in the area [1,2,4,17,22,28,39,43,45,46] provoke deep questions concerning the connection between birational geometry, dervied categories, and arithmetic geometry; this makes me incredibly excited about the prospect of diving further into my own recent projects via probing questions. With this in mind, one of the questions motivating my research is the following:

> **Question.** To what extent can $\mathrm{D}^{\mathrm{b}}(X)$ be used to determine arithmetic properties of X?

As a concrete example of this, consider two strictly Calabi-Yau manifolds X and Y defined over an arbitrary field k that are derived equivalent, that is $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(Y)$. In a similar vein as the question addressed in [2], one can ask if it is possible for only one of X or Y to admit a rational point. *Homological Projective Duality* [38] has been an incredibly useful tool for producing non-birational Fourier-Mukai partners, so one would hope that it will be equally as a useful in producing Fourier-Mukai partners with disparate arithmetic properties.

4.1 *Future Directions:* Toric Degenerations

In the 1990's, Batyrev [11] introduced the notion of a *small toric degeneration* of a manifold M defined over $\mathbb{C}$. Producing a *toric degeneration* of a manifold M is, in essence, approximating M via well understood, albeit likely mildly singular, toric varieties. I would like to classify the rationality properties of (forms of) small toric degenerations of Fano 3-folds of Picard rank > 1.

Smooth Fano 3-folds have been completely classified, [31] and recent work of Galkin [25] classifies 3-dimensional nontoric smooth Fano varieties that admit such degenerations. In general, given a non-toric fano 3-fold M for which a small toric degeneration exists, the toric varieties that form the degeneration may be singular. However, these singularities lie on the boundaries of the toric varieties, i.e. not in the dense open tori themselves. The rationality of the tori dictates the rationality of each toric variety, so we can easily determine the rationality status of the small toric degeneration. Rational tori have been classified [34,35,40] up to dimension four, so everything needed for my desired classification already exists in the literature; it is just a matter of the right person being able to put these things together.

It is well known that complex Fano curves and surfaces are rational, and that dimension three is where the rationality questions regarding complex Fano varieties become interesting. For example, cubic three-folds in $\mathbb{P}^4$ are complex Fano varieties that were shown to be nonrational by Clemens and Griffiths [22]. Recent work of Hassett and Tschinkel [28] indicates that this classification may yield interesting examples of geometrically rational smooth Fano 3-folds that are non-rational. The case of Picard rank one is essentially complete due to [28,39], so I plan to look at Fano 3-folds of Picard rank > 1.

References

[1] Achter, J.D.; Casalaina-Martin, S; Vial, C. *Derived equivalent threefolds, algebraic representatives, and the coniveau filtration*. Math. Proc. Cambridge Philos. Soc. 167 (2019), no. 1, 123–131. MR3968824

[2] Addington, N; Antieau, B; Frei, S; Honigs, K. *Rational points and derived equivalence*. arXiv:1906.02261.

[3] Alexeev, V; Thompson, A. *ADE surfaces and their moduli*. arXiv:1712.07932.

[4] Auel, A; Bernardara, M. *Semiorthogonal decompositions and birational geometry of del Pezzo surfaces over arbitrary fields*. Proc. Lond. Math. Soc. (3) 117 (2018), no. 1, 1–64

[5] *Brauer groups and obstruction problems*. Moduli spaces and arithmetic. Edited by Asher Auel, Brendan Hassett, Anthony Várilly-Alvarado and Bianca Viray. Progress in Mathematics, 320. Birkhäuser/Springer, Cham, 2017. ix+247 pp. ISBN: 978-3-319-46851-8; 978-3-319-46852-5 MR3587845

[6] Ballard, M.R.; Duncan, A; McFaddin, P.K. *On derived categories of arithmetic toric varieties*. arXiv:1709.03574.

[7] Ballard, M. R.; Duncan, A; McFaddin, P.K. *Derived categories of centrally symmetric toric Fano varieties*. arXiv:1812.09392.

[8] Ballard, Matthew R.; Lamarche, A. *On derived categories and rational points for a class of toric Fano varieties*. In preparation.

[9] Ballard, M.R.; Duncan, A; Lamarche, Alicia; McFaddin, P.K. *Coflasque resolutions and Brauer group invisibility*. In preparation.

[10] Ballard, M. R.; Duncan, A; Lamarche, A; McFaddin, P.K. *Exceptional collections, rationality, and rational points*. In preparation.

[11] Batyrev, V. *Toric degenerations of Fano varieties and constructing mirror manifolds*. The Fano Conference, 109–122, Univ. Torino, Turin, 2004. MR2112571

[12] Batyrev, V; Blume, M. *The functor of toric varieties associated with Weyl chambers and Losev-Manin moduli spaces*. Tohoku Math. J. (2), 63(4):581–604, 2011.

[13] Batyrev, V; Blume, M. *On generalisations of Losev-Manin moduli spaces for classical root systems*. Pure Appl. Math. Q. 7 (2011), no. 4, Special Issue: In memory of Eckart Viehweg, 1053–1084. MR2918154

[14] Beilinson, A.A. *Coherent sheaves on $\mathbb{P}^n$ and problems in linear algebra*. Funct Anal Its Appl (1978) 12: 214. https://doi.org/10.1007/BF01681436

[15] Berghoff, M. *Wonderful Compactifications in Quantum Field Theory*. arXiv:1411.5583

[16] Bernardara, M; Bolognesi, M. *Categorical representability and intermediate Jacobians of Fano threefolds*. Derived categories in algebraic geometry, 1–25, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012. MR3050698

[17] Blunk, M. *Del Pezzo surfaces of degree 6 over an arbitrary field.* J. Algebra 323 (2010), no. 1, 42–58. MR2564828

[18] Brion M. (1989) *Spherical Varieties an Introduction.* In: Kraft H., Petrie T., Schwarz G.W. (eds) Topological Methods in Algebraic Transformation Groups. Progress in Mathematics, vol 80. Birkhäuser Boston

[19] Bondal, O. *Reconstruction of a variety from the derived category and groups of autoequivalences.* arXiv:alg-geom/9712029.

[20] Brown, M; Shipman, I. *The McKay correspondence, tilting equivalences, and rationality.* arXiv:1312.3918

[21] Castravet, A-M; Tevelev, J. *Derived category of moduli of pointed curves - I.* arXiv:1708.06340.

[22] Clemens, H.; Griffiths, P. *The intermediate Jacobian of the cubic threefold*, Ann. of Math. 95 (1972), 73–100.

[23] Colliot-Thélène, J-L. *Résolutions flasques des groupes linéaires connexes.* J. Reine Angew. Math. 618 (2008), 77–133.

[24] Costa, L.; Miró-Roig, M.R. *Frobenius splitting and Derived category of toric varieties.* arXiv:1006.5315.

[25] Galkin, S. *Small toric degenerations of Fano threefolds.* arXiv:1809.02705.

[26] Gonzales, Richard; Pech, Clelia; Perrin, Nicolas; Samokhin,Alexander *Geometry of horospherical varieties of Picard rank one.* arXiv:1803.05063.

[27] Gross, M; Siebert, B.*From real affine geometry to complex geometry.* Ann. of Math. (2) 174 (2011), no. 3, 1301–1428. MR2846484

[28] Hassett, B; Tschinkel, Y. *Rationality of complete intersections of two quadrics.* arXiv:1903.08979.

[29] Hosono, S.; Takagi, H. *Determinantal quintics and mirror symmetry of Reye congruences.* Comm. Math. Phys., 329(3):1171–1218, 2014. Also arXiv:1208.1813.

[30] Hosono S.; Takagi, H. *Double quintic symmetroids, Reye congruences, and their derived equivalence.* J. Differential Geom., 104(3):443– 497, 2016. Also arXiv:1302.5883.

[31] Iskovskikh, V. A.; Prokhorov, Yu. G. (1999), *Fano varieties*, in A. N. Parshin; I. R. Shafarevich (eds.), Algebraic Geometry, V. Encyclopedia Math. Sci., 47, Springer-Verlag, pp. 1–247, ISBN 3-540-61468-0, MR 1668579

[32] Kawamata, Y. *Derived categories of toric varieties.* Michigan Math. J. 54 (2006), no. 3, 517–535. MR2280493

[33] Kawamata, Y. *Derived categories of toric varieties II.* Michigan Math. J. 62 (2013), no. 2, 353–363. MR3079267

[34] Kunyavskii, B. E. *Three-dimensional algebraic tori.* In Investigations in number theory (Russian), pages 90–111. Saratov. Gos. Univ., Saratov, 1987. Translated in Selecta Math. Soviet. 9 (1990), no. 1, 1–21.

[35] Kunyavskii, B. E. *Algebraic tori - thirty years after.* arXiv:0712.4061.

[36] Kuznetsov, A. *Derived categories view on rationality problems.* In Rationality problems in algebraic geometry, volume 2172 of Lecture Notes in Math., pages 67–104. Springer, Cham, 2016.

[37] Kuznetsov, A. *Derived categories of Fano threefolds.* arXiv:0809.0225.

[38] Kuznetsov, A. *Homological Projective Duality.* arXiv:math/0507292.

[39] Kuznetsov, A; Prokhorov, Y. *Rationality of prime Fano 3-folds over nonclosed fields.* In progress.

[40] Lemire, N. *Four-Dimensional Algebraic Tori.* arXiv:1511.00315.

[41] Losev, A.; Manin, Y. *New moduli spaces of pointed curves and pencils of flat connections.* Michigan Math. J. 48 (2000), no. 1, 443–472.

[42] Okawa, S. *Semi-orthogonal decomposability of the derived category of a curve.* Adv. Math. 228 (2011), no. 5, 2869–2873. MR2838062

[43] Raedschelders, T. *Non-split Brauer-Severi varieties do not admit full exceptional collections.* arXiv:1605.09216.

[44] Serre, J-P. *Cohomological invariants, Witt invariants, and trace forms.* Notes by Skip Garibaldi. Univ. Lecture Ser., 28, Cohomological invariants in Galois cohomology, 1–100, Amer. Math. Soc., Providence, RI, 2003.

[45] Tabuada, G; Van den Bergh, M. *Noncommutative motives of separable algebras*, arXiv:1411.7970 (2014).

[46] Vial, C. *Exceptional collections, and the Néron-Severi lattice for surfaces.* Adv. Math. 305 (2017), 895–934. MR3570150

[47] Voskresenskii, V.E.; . Klyachko, A.A. *Toroidal Fano varieties and root systems*, Izv. Akad. Nauk SSSR Ser. Mat., 48:2 (1984), 237–263; Math. USSR-Izv., 24:2 (1985), 221–244